

On global geodesic mappings of n -dimensional surfaces of revolution

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Abstract

In this paper we study geodesic mappings of n -dimensional surfaces of revolution. From the general theory of geodesic mappings of equidistant spaces we specialize to surfaces of revolution and apply the obtained formulas to the case of rotational ellipsoids. We prove that such n -dimensional ellipsoids admit non trivial smooth geodesic deformations onto n -dimensional surfaces of revolution, which are generally of a different type.

Keywords: geodesic mapping, pseudo-Riemannian spaces, Riemannian spaces, surface of revolution

subclass: 53B20; 53B21; 53B30; 53C22; 53C25; 53C40

1 Introduction

We consider geodesic mappings of smooth, closed n -dimensional surfaces of revolution. Such mappings may be constructed with the aid of embedding into an $(n+1)$ -dimensional auxiliary space. In [10] the embedding space of an ellipsoid is equipped once with the euclidian metric, once with a different one. Either metric induces a metric on the ellipsoid, and by a convenient choice of the metrics of the embedding space, the induced metrics on the ellipsoids are geodesically equivalent, i. e. have the same geodesics.

Another method is to leave the euclidian metric of the embedding space invariant and to deform the ellipsoid. On each of the deformed (hyper)surfaces the ambient euclidian metric induces a metric. In [17] a one-parameter family of smooth deformations is introduced, which maps geodesics to geodesics. In other words, they are geodesic mappings between different (hyper)surfaces, embedded into the $n + 1$ -dimensional euclidian space E_{n+1} .

On the other hand, all the deformed surfaces being diffeomorphic to each other, we can forget the embedding and consider them as one and the same manifold with the global topology of the n -dimensional sphere S_n . From this point of view our approach generates a one-parameter family of geodesically equivalent metrics on this manifold. For an ellipsoid the metrics generated in [10] are special cases of this family.

Concretely we take for simplicity a 2-dimensional rotational ellipsoid in E_3 , deform it in a certain way according to [17], and construct metrics on the deformed surfaces, induced by the euclidian metric of the embedding space. When we pull back these metrics to the original ellipsoid, we have a one-parameter family of geodesically equivalent metrics on one surface. At first instance, this construction is a realization of geodesic mappings between different surfaces of revolution, further, when the metrics of the deformed surfaces are pulled back, the foregoing deformation gives an intuitive impression of the

modified, geodesically equivalent metrics. The generalization to n -dimensional surfaces of revolution is straightforward.

In the first part (section 2) of the present article we give a brief introduction into the theory of geodesic mappings of equidistant spaces. Geodesic mappings are diffeomorphisms $V_n \rightarrow \bar{V}_n$ between n -dimensional (pseudo-) Riemannian spaces that take any geodesic of V_n into a geodesic of \bar{V}_n . Equidistant spaces are manifolds which are foliated by $(n-1)$ -dimensional subspaces. In sections 3 and 4 we specialise to n -dimensional surfaces of revolution with rotational symmetry around a certain axis.

In the main part (section 5 and 6) we study rotational ellipsoids and the class of geodesic mappings introduced in [17] as examples. We find that these mappings act nontrivially and map ellipsoids onto surfaces of revolution of a different kind, the explicit description of which is found.

2 Geodesic mappings of equidistant spaces.

General problems of the theory of geodesic mappings have been studied by T. Levi-Civita [9] and many others [1, 3, 8, 17, 18, 20, 23, 24, 25]. It is known that a necessary and sufficient condition for the existence of geodesic mappings between n -dimensional Riemannian manifolds $V_n(M, g)$ and $\bar{V}_n(M, \bar{g})$, is $(\bar{\nabla} - \nabla)_X X = 2\nabla_X \psi X$, $\forall X \in TV_n$, where ψ is a function on M and ∇ and $\bar{\nabla}$ are the Levi-Civita connections of the metric tensors g and \bar{g} , respectively. In local coordinates this “coordinate-free” formula introduced above is equivalent to the Levi-Civita equation [3, 9, 16, 17, 18, 20]

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \delta_i^h \psi_j,$$

where $\psi_i(x)$ are the components of the gradient $\nabla\psi$, Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of V_n and \bar{V}_n .

Geodesic mappings of equidistant spaces were studied by N.S. Sinyukov [22, 23] and, in the following, by many other authors, for example [4, 5, 6, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 26].

Here we present a type of geodesic mappings, introduced in [16, 23]. Assume the equidistant spaces V_n and \bar{V}_n , which have metrics of the following shape [4, 5]:

$$ds^2 = a(w) dw^2 + b(w) d\sigma^2 \quad (1)$$

and

$$d\bar{s}^2 = \frac{p a(w)}{(1 + qb(w))^2} dw^2 + \frac{p b(w)}{1 + qb(w)} d\sigma^2, \quad (2)$$

where a and b are differentiable functions of the variable w , p and q are real parameters, and $d\sigma^2$ is a metric of an $(n-1)$ -dimensional (pseudo-) Riemannian space \tilde{V}_{n-1} .

These spaces V_n and \bar{V}_n have common geodesics, i.e. $V_n \rightarrow \bar{V}_n$ is a geodesic mapping and the function in the Levi-Civita equation is $\psi = -\frac{1}{2} \ln |1 + q \cdot b(w)|$.

If $q b'(w) \neq 0$, the mapping is non trivial (i.e. is not affine). In [4, 5, 18] this is shown by comparing the Levi-Civita connections of both spaces.

3 n -dimensional surfaces of revolution

It is well known that surfaces of revolution and their n -dimensional generalizations are equidistant spaces. Geodesic mappings of such surfaces are studied in the papers by J. Mikeš [15, 16, 17]. The

results obtained there guarantee the existence of nontrivial geodesic mappings of surfaces of revolution of class C^1 , including surfaces which are homeomorphic to the n -dimensional sphere.

Let \mathcal{S}_n denote a surface of revolution of class C^1 in the $(n+1)$ -dimensional Euclidian space E_{n+1} . In the cartesian coordinates system (x^1, \dots, x^{n+1}) the surface \mathcal{S}_n can be determined by the conditions:

$$x^i = r(w) u^i \quad (i = 1, 2, \dots, n), \quad x^{n+1} = z(w), \quad (3)$$

where u^i are the coordinates of the standard unit sphere

$$S_{n-1} = \{(u^1, u^2, \dots, u^n) \mid (u^1)^2 + (u^2)^2 + \dots + (u^n)^2 = 1\},$$

$w \in [w_1, w_2]$ is a parameter of horizontal type, w_1, w_2 are fixed values (or $-\infty$ and $+\infty$, respectively), $r(w), z(w) \in C^1[w_1, w_2]$, $r(w) > 0$ for every $w \in (w_1, w_2)$.

If $r(w_1) = r(w_2) = 0$ then \mathcal{S}_n is homeomorphic to the n -dimensional sphere; the points corresponding to w_1 and w_2 are called *poles*. In the case $r(w_1) = 0$ and $r(w_2) \neq 0$ the surface \mathcal{S}_n is homeomorphic to E_n or an n -dimensional disk. If $r(w_1) = r(w_2) \neq 0$ and $z(w_1) = z(w_2)$, then the surface \mathcal{S}_n is homeomorphic to the n -dimensional torus.

In these coordinates the metric on \mathcal{S}_n has the form

$$ds^2 = (r'^2(w) + z'^2(w)) dw^2 + r^2(w) d\sigma^2, \quad (4)$$

where $d\sigma^2$ is a metric of the sphere S_{n-1} .

Remark. In the papers [15, 16, 17], $r'^2 + z'^2 = 1$ is assumed, so that the parameter w denotes the arc length along the “meridians”. For \mathcal{S}_n to be smooth at the poles, corresponding to the parameter values w_* , the derivatives of r and z must satisfy $|r'(w_*)| = 1$ and $z'(w_*) = 0$.

4 Geodesic mappings of n -dimensional surfaces of revolution

The metric (4) of a surface of revolution appears as a special case of the metric (1) of a general equidistant space with

$$a(w) = r'^2(w) + z'^2(w) \quad \text{and} \quad b(w) = r^2(w). \quad (5)$$

In the next section we will construct geodesic mappings to spaces \bar{V}_n , according to section 2, with a metric of the type,

$$d\bar{s}^2 = \bar{a}(w) dw^2 + \bar{b}(w) d\sigma^2, \quad (6)$$

$$\bar{a}(w) = p \frac{r'^2 + z'^2}{(1 + qr^2)^2} \quad \text{and} \quad \bar{b}(w) = p \frac{r^2}{1 + qr^2}, \quad (7)$$

where p, q are constants and, evidently $\bar{a}(w), \bar{b}(w) \neq 0$ and $\bar{a}(w), \bar{b}(w) < \infty$, when

$$p \neq 0; \quad r'^2(w) + z'^2(w) \neq 0; \quad 1 + qr^2(w) \neq 0 \quad \forall w \in [w_1, w_2]. \quad (8)$$

When $q = 0$, the space \bar{V}_n is homothetic (with the coefficient $p \neq 0$) to the surface \mathcal{S}_n .

From the formulas (6) and (7) follows that the set of spaces \bar{V}_n , to which \mathcal{S}_n maps geodesically, depends on two parameters p and q .

The parameter p corresponding to homothetic transformations can be taken positive, q is restricted by the condition $1 + qr^2(w) \neq 0$ for every $w \in [w_1, w_2]$.

In the case

$$1 + qr^2(w) < 0$$

the metric of \bar{V}_n has Minkowski signature and this space can not be realized as surface on a Euclidean space. In this case $r(w) \neq 0$ for all w and \mathcal{S}_n is homeomorphic to an n -dimensional cylinder, and

$$q \in \left(-\infty, -\max_{w \in [w_1, w_2]} r^{-2}(w) \right).$$

On the other hand if

$$1 + qr^2(w) > 0$$

the metric of \bar{V}_n is positive definite and q lies in the interval $q \in \left(-\min_{w \in [w_1, w_2]} r^{-2}(w), +\infty \right)$.

In this case the metric (6) might but need not be realised on some surface of revolution. The ellipsoids considered in the remainder of this paper are mapped to closed surfaces of revolution.

5 A special class of geodesic mappings

We suppose that the n -dimensional surface of revolution \mathcal{S}_n , which is defined by equations (3), is mapped on an n -dimensional surface of revolution $\bar{\mathcal{S}}_n$, which is defined in a similar manner:

$$x^i = \bar{r}(w)u^i, \quad (i = 1, 2, \dots, x^n), \quad x^{n+1} = \bar{z}(w), \quad (9)$$

where \bar{r} and \bar{z} are functions of the same parameter w as before. In the following we restrict our attention to 2 dimensional surfaces generated by the rotation of a curve around an axis, denoted by z . The generalization to higher-dimensional hypersurfaces is straightforward. The curve is formulated in parametric form $r(w)$, $z(w)$, where r is a radial variable and the parameter w is the arc length. The relation between $z(w)$ and some suitably given function $r(w)$ is given by

$$z(w) = \int_{w_1}^w e(\tau) \sqrt{1 - r'^2(\tau)} d\tau, \quad (10)$$

where $e(\tau)$ can assume the values ± 1 on different pieces of the curve. In the following we assume $e = +1$ throughout the entire curve, so that z grows monotonically with the length w and the surface is convex. In [17] this case is denoted by the terminus “simple surface of revolution”. w runs from w_1 to w_2 , $w_2 - w_1$ is the total length of the curve. We assume a closed surface, i.e. $r(w_1) = r(w_2) = 0$. For the sake of smoothness at the rotation poles we assume also $|\frac{dr}{dw}| = 1$ at w_1 and w_2 , then $\frac{dz}{dw} = 0$ at the poles follows automatically from (10).

In [17] the curve $(r(w), z(w))$ is acted upon by the following one-parameter family of transformations f : $\mathcal{S}_n \rightarrow \bar{\mathcal{S}}_n$, labeled by a . It is defined by the following action on the functions $r(w)$ and $z(w)$:

$$\bar{r}(w) = \frac{r(w)}{\sqrt{1 + ar^2(w)}}, \quad \bar{z}(w) = \int_{w_1}^w \sqrt{\frac{1 + ar^2(\tau) - r'^2(\tau)}{(1 + ar^2(\tau))^3}} d\tau. \quad (11)$$

The parameter w is the same as before, therefore it is *not* the length parameter of the curve (\bar{r}, \bar{z}) . The conditions of smoothness $\bar{r} = 0$, $\frac{d\bar{r}}{dw} = \pm 1$ and $\frac{d\bar{z}}{dw} = 0$ at the poles $w = w_1$ and $w = w_2$ are satisfied, provided they are satisfied for r .

It is further shown in [15] from comparison of the metrics on the surfaces generated by rotation of the curves (r, z) and (\bar{r}, \bar{z}) that these mappings leave geodesics invariant. The metric of \mathcal{S}_n of the form (4),

$$ds^2 = dw^2 + r^2(w) d\sigma^2, \quad (12)$$

is mapped to

$$d\bar{s}^2 = (\bar{r}'^2(w) + \bar{z}'^2(w)) dw^2 + \bar{r}^2(w) d\sigma^2. \quad (13)$$

In terms of r and z this is explicitly

$$d\bar{s}^2 = \frac{dw^2}{[1 + ar^2(w)]^2} + \frac{r^2(w) d\sigma^2}{1 + ar^2(w)}. \quad (14)$$

The relation between the metrics (12) and (14) is of the same type as between (1) and (2). In the case of a 2-dimensional surface of revolution embedded into E_3 , σ is simply the angle around the rotation axis.

From the point of view of the embedding into 3-dimensional euclidian space, the transformation f is a smooth deformation. On the other hand, the formulation (14) suggests to consider the original and the transformed surface as one and the same manifold, coordinatized by w and the rotation angle σ , and f as a transformation of metrics. Then from (14) it follows that both the metrics ds and $d\bar{s}$ have the same geodesics.

6 Application to rotational ellipsoids

In the foregoing section we have seen a class of nontrivial geodesic mappings between smooth surfaces of revolution, which are homeomorphic to a sphere. Now we take as a concrete example a rotational ellipsoid, embedded into the 3-dimensional Euclidian space, and investigate its deformation by the considered geodesic mappings. This is done in a local coordinate patch, covering one half of the surface. Rather than in terms of the arc length w we formulate it in terms of the angular variable φ ,

$$r(\phi) = k \sin \varphi, \quad z(\varphi) = 1 - \cos \varphi. \quad (15)$$

The squared element of the arc length is

$$dw^2 = dr^2 + dz^2 = (k^2 \cos^2 \varphi + \sin^2 \varphi) d\varphi^2. \quad (16)$$

We choose $w_1 = w(\varphi = 0) = 0$, so that the origin of φ and the arc length coincide, then $w_2 = w(\varphi = \pi)$ is half of the circumference of the ellipse. The condition $r(w_1) = r(w_2) = 0$ is fulfilled and

$$\frac{dr}{dw} = \frac{dr}{d\varphi} \frac{d\varphi}{dw} = \frac{k \cos \varphi}{\sqrt{k^2 \cos^2 \varphi + \sin^2 \varphi}}, \quad (17)$$

so also $\frac{dr}{dw}(w_1) = 1$ and $\frac{dr}{dw}(w_2) = -1$ are satisfied.

To carry out the transformation (11), we refer to the parameter ϕ rather than to w for calculational convenience. What is important, is to use the same parameter for (r, z) and (\bar{r}, \bar{z}) . In terms of φ we have

$$\bar{r}(\varphi) = \frac{k \sin \varphi}{\sqrt{1 + ak^2 \sin^2 \varphi}} \quad (18)$$

and

$$\bar{z}(\varphi) = \int_0^\varphi \sqrt{\frac{1 + ar^2(\varphi') - r'^2(\varphi')}{(1 + ar^2(\varphi'))^3}} \frac{dw}{d\varphi'} d\varphi'. \quad (19)$$

Note that here and in the following r' means always the derivative with respect to w , even when written as function of w , so $r'(\varphi)$ is $\frac{dr}{d\varphi} \frac{d\varphi}{dw}$ as function of φ .

Explicitly we find

$$\bar{r}'(\varphi) = \frac{k \cos \varphi}{(k^2 \cos^2 \varphi + \sin^2 \varphi)^{\frac{1}{2}} (1 + ak^2 \sin^2 \varphi)^{\frac{3}{2}}}, \quad (20)$$

the maximal value of \bar{r} , $\bar{r}_{\max} = \frac{k}{\sqrt{1+ak^2}}$, occurs at $\varphi = \frac{\pi}{2}$, like for the original ellipsoid.

To find out what kind of curve is $(\bar{r}(\varphi), \bar{z}(\varphi))$, we should solve the integral (20) explicitly, which is complicated. So we consider rather the derivative $\frac{d\bar{z}}{d\bar{r}}$, which gives a differential equation for the curve, and eliminate the parameter. This is done in several steps:

First we express $\frac{d\bar{z}}{d\bar{r}}$ in the form $\frac{d\bar{z}}{d\varphi} / \frac{d\bar{r}}{d\varphi}$. From (18) and (19) we get

$$\frac{d\bar{r}}{d\varphi} = \frac{k \cos \varphi}{(1 + ak^2 \sin^2 \varphi)^{\frac{3}{2}}}$$

and

$$\frac{d\bar{z}}{d\varphi} = \sqrt{\frac{1 + ar^2(\varphi) - r'^2(\varphi)}{(1 + ar^2(\varphi))^3}} \frac{dw}{d\varphi}.$$

Then from the definitions (15) and the explicit equation of the ellipse

$$(1 - z)^2 + \frac{r^2}{k^2} = 1$$

we express $\sin \varphi$ and $\cos \varphi$ in terms of r and find

$$\frac{d\bar{z}}{d\bar{r}} = \frac{r\sqrt{1 + ak^4 + ar^2 - a^2k^2r^2}}{k\sqrt{k^2 - r^2}} \quad (21)$$

in terms of r .

Now we insert the inverse of (18),

$$r = \frac{\bar{r}}{\sqrt{1 - a\bar{r}^2}} \quad (22)$$

to express this derivative in terms of \bar{r} ,

$$\frac{d\bar{z}}{d\bar{r}} = \frac{\bar{r} \sqrt{\frac{1}{k^2} + ak^2 - a(1 + ak^2)\bar{r}^2}}{\sqrt{1 - a\bar{r}^2} \sqrt{k^2 - (1 + ak^2)\bar{r}^2}}. \quad (23)$$

At last, for a direct comparison with the corresponding differential equation for an ellipse,

$$\frac{dz}{dr} = \frac{r}{k\sqrt{k^2 - r^2}}, \quad (24)$$

we carry out a scale transformation

$$\hat{r} = \bar{r}\sqrt{1 + ak^2}, \quad \hat{z} = \bar{z}\sqrt{1 + ak^2}, \quad (25)$$

so that the maximal value of \hat{r} is equal to k , like the maximal value of r in the case of the ellipse and the radial extensions of both surfaces are the same. In terms of these variables, finally,

$$\frac{d\hat{z}}{d\hat{r}} = \frac{\hat{r}}{k\sqrt{k^2 - \hat{r}^2}} \cdot \sqrt{\frac{1 + ak^2(k^2 - \hat{r}^2)}{1 + a(k^2 - \hat{r}^2)}}. \quad (26)$$

From this we can see that the transformed curve is of a different type than an ellipse, the differential equation of the corresponding ellipse is modified by the factor right to the dot in (26). At the maximal values of the radial variables, i. e. at the “equator”, both the derivatives $\frac{dz}{dr}$ for the ellipse and $\frac{d\hat{z}}{d\hat{r}}$ for the deformed curve go to infinity, corresponding to the fact that r and σ provide only a local chart for one half of the surface.

An interesting feature of these transformations is that they leave circles ($k = 1$) invariant (up to a scale factor $\sqrt{1+a}$), in the limit of a large transformation parameter a the modification factor in (26) goes to k and the (rescaled) transformed curve approaches a circle.

The metric of the resulting surface of revolution,

$$ds^2 = \left(1 + \frac{d\hat{z}^2}{d\hat{r}^2}\right) d\hat{r}^2 + \hat{r}^2 d\sigma^2,$$

is

$$ds^2 = \frac{k^2 + ak^4 + \left(\frac{1}{k^2} - ak^2 - 1\right) \hat{r}^2}{(k^2 - \hat{r}^2)(1 + ak^2 - a\hat{r}^2)} d\hat{r}^2 + \hat{r}^2 d\sigma^2. \quad (27)$$

This form of the metric in terms of \hat{r} is local and applies only to the lower or the upper half of the surface. It can be generalized without problems to higher dimensions, when the circles with constant \hat{z} are replaced by higher-dimensional spheres. Then $d\sigma$ has only to be replaced by the solid angle element $d\Omega$ of the corresponding dimension.

This metric can be pulled back to the original ellipsoid by simply expressing \hat{r} in terms of r ,

$$ds^2 = (1 + ak^2) \left[\frac{k^2 + \left(\frac{1}{k^2} - 1\right) r^2}{(k^2 - r^2)(1 + ar^2)^2} dr^2 + \frac{r^2}{1 + ar^2} d\sigma^2 \right], \quad (28)$$

whereas the metric on the original ellipsoid is

$$ds^2 = \frac{k^2 + \left(\frac{1}{k^2} - 1\right) r^2}{k^2 - r^2} dr^2 + r^2 d\sigma^2. \quad (29)$$

For an explicit expression of the deformed surfaces, we calculate the equations of the “meridians” in the form $\hat{z}(r)$, where \hat{z} and \hat{r} are cartesian coordinates of a cross-section through the rotation axis. For this purpose we integrate (26), from now on we drop the hats on r and z . We begin with the substitution

$$\sin^2 \phi = \frac{k^2 - r^2}{k^2 - \frac{1}{a} - r^2}. \quad (30)$$

Then

$$z(r) = -\frac{1}{k\sqrt{a}} \int_{\phi(0)}^{\phi(r)} \frac{\sqrt{1 - (1 - k^2) \sin^2 \phi}}{\cos^2 \phi} d\phi, \quad (31)$$

where

$$\phi(0) = \arcsin \sqrt{\frac{ak^2}{1 + ak^2}} \quad \text{and} \quad \phi(r) = \arcsin \sqrt{\frac{a(k^2 - r^2)}{1 + ak^2 - ar^2}}.$$

Integrating (31) by parts gives

$$\begin{aligned} \int_{\phi(0)}^{\phi(r)} \sqrt{1 - (1 - k^2) \sin^2 \phi} \frac{d\phi}{\cos^2 \phi} &= \sqrt{1 - (1 - k^2) \sin^2 \phi} \tan \phi \Big|_{\phi(0)}^{\phi(r)} \\ &- \frac{1}{(1 - k^2)} \int_{\phi(0)}^{\phi(r)} \sqrt{1 - (1 - k^2) \sin^2 \phi} d\phi + \frac{1}{(1 - k^2)} \int_{\phi(0)}^{\phi(r)} \frac{d\phi}{\sqrt{1 - (1 - k^2) \sin^2 \phi}} \end{aligned}$$

where the last two integrals are the standard elliptic integrals of the second and first kind [2]

$$E(\phi, \kappa) = \int_0^\phi \sqrt{1 - \kappa^2 \sin^2 \phi} d\phi \quad \text{and} \quad F(\phi, \kappa) = \int_0^\phi \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}}.$$

Inserting back r gives finally

$$\begin{aligned}
z(r) = & -\frac{\sqrt{k^2 - r^2}}{k} \sqrt{\frac{1 + ak^4 - ak^2 r^2}{1 + ak^2 - ar^2}} + \sqrt{\frac{1 + ak^4}{1 + ak^2}} + \\
& \frac{1}{\sqrt{a} k(1 - k^2)} \left[E \left(\arcsin \sqrt{\frac{k^2 - r^2}{k^2 + \frac{1}{a} - r^2}}, \sqrt{1 - k^2} \right) - E \left(\arcsin \frac{k}{\sqrt{k^2 + \frac{1}{a}}}, \sqrt{1 - k^2} \right) \right. \\
& \left. - F \left(\arcsin \sqrt{\frac{k^2 - r^2}{k^2 + \frac{1}{a} - r^2}}, \sqrt{1 - k^2} \right) + F \left(\arcsin \frac{k}{\sqrt{k^2 + \frac{1}{a}}}, \sqrt{1 - k^2} \right) \right] \quad (32)
\end{aligned}$$

For small values of the parameter a we have in linear approximation

$$z(r) = 1 - \frac{\sqrt{k^2 - r^2}}{k} - a \cdot \frac{k^2(1 - k^2)}{6} + a \cdot \frac{1 - k^2}{6k} (k^2 - r^2)^{\frac{3}{2}} + O(a^2).$$

From this we see that the z coordinate of the “equator”, where $r = k$, is shifted by $-a \cdot \frac{k^2(1 - k^2)}{6}$, so according to the sign of a and $1 - k^2$ the surface becomes “compressed” or “elongated”.

7 Summary

We have considered two aspects of geodesic mappings of ellipsoids. (26) and (32) describe the transformations as deformations in E_3 . An interesting property is that on a sphere as a special case of an ellipsoid these transformations act as identity, whereas they act highly non trivially on general ellipsoids. In the limit of large transformation parameters the transformed surfaces approach a sphere as limiting surface.

The second aspect, represented by (28) and (29), concerns geodesic transformations of the metric on a manifold homeomorphic to the sphere, in accordance with [27], where it is shown by application of a classical theorem by Dini [7] that there is (up to homothety) a one-parameter family of geodesically equivalent metrics on S_2 .

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